QUADRATIC EQUATION

GENERAL THEORY OF EQUATIONS

Polynomial equation: A polynomial which is equal to zero is called a polynomial equation. For example, $2x + 5 = 0$, $x^2 - 2x + 5 = 0$, $2x^2 - 5x^2 + 1 = 0$ etc. are polynomial equations.

Root of a polynomial equation: If $f(x) = 0$ is a polynomial equation and $f(\alpha) = 0$, then α is called a root of the polynomial equation $f(x) = 0$.

Theorem: Every equation $f(x) = 0$ of n^{th} degree has exactly n roots.

Factor Theorem: If α is a root of equation $f(x) = 0$, then the polynomial $f(x)$ is exactly divisible by $x - \alpha$ (i.e., remainder is zero).For example, $x^2 - 5x + 6 = 0$ is divisible by $x - 2$ because 2 is the root of the given equation.

Multiplicity of a Root: If α is a root of the polynomial equation $f(x) = 0$ then α is called a root of multiplicity r if $(x - \alpha)^r$ divides $f(x)$.
For example: In the equation $(x + 1)^4 (x - 2)^2 (2x - 6) = 0$ the root $-1, 2, 3$ are of multiplicity 4, 2, and 1 respectively.

Linear Equation: A linear equation is 1st degree equation. It has only one root. Its general form is $ax + b = 0$ and root is $-b/a$.

Quadratic Equation: A quadratic equation in one variable has two and only two roots. The general form of a quadratic equation is $ax^2 + bx + c = 0$. This equation has two & only two roots,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If X_1 and X_2 are the roots, then

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \ \& \ x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

NATURE OF ROOTS:

* The term $(b^2 - 4ac)$ is called the discriminant of the quadratic equation $aX^2 + bX + c$ and is denoted by D.

Rules:

1. if $D > 0$, X_1 & X_2 are real and unequal
2. if $D = 0$, X_1 & X_2 are real and equal
3. if D is a perfect square, X_1 & X_2 are rational and unequal
4. if $D < 0$, X_1 & X_2 are imaginary, unequal, and conjugates of each other.

If X_1 and X_2 are the two roots of $aX^2 + bX + C = 0$
then sum of roots = $X_1 + X_2 = -b/a$
and product of roots = $X_1 X_2 = c/a$

Deductions:

1. If $b = 0$, $X_1 + X_2 = 0$ or $X_1 = -X_2,$
   ***Converse*:** If roots of given equation are equal in magnitude but opposite in sign, then b = 0
2. If c = 0, one of the roots will be zero & vice versa
3. if $c = a$, $X_1 = 1/X_2$
   Converse: If roots of given equation are reciprocal of each other, then c = a

FORMATION OF EQUATION FROM ROOTS:

∗. If X_1 and X_2 are the two roots then $(X - X_1)(X - X_2) = 0$ is the required equation

∗ If $(X_1 + X_2)$ and X_1, X_2 are given the equation is $X^2 - (X_1 + X_2) X + X_1 X_2 = 0$
$\Rightarrow X^2 - SX + P = 0$ where S = sum of roots, P = product of roots.

TRANSFORMATION OF EQUATIONS:

Let the given equation be $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots + a_n = 0$ having roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \ldots.\alpha_n$.
Some simple transformations of this equation are given below :

Sr. No	Required nature of roots	Reqd roots	Reqd Transformation	Resulting Equation
1	Opposite sign	$-\alpha_1, -\alpha_2, -\alpha_3, \ldots.$	Put $x = -x$	$a_0 x^n - a_1 x^{n-1} + a_2 x^{n-2} - \ldots. + (-1)^n a_n = 0$
2	Reciprocal	$1/\alpha_1, 1/\alpha_2, 1/\alpha_3, ..$	Put $x = 1/x$	$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots\ldots + a_0 = 0$
3	Reciprocal & opp. sign	$-1/\alpha_1, -1/\alpha_2, -1/\alpha_3 \ldots\ldots$	Put $x = -1/x$	$a_n x^n - a_{n-1} x^{n-1} + a_{n-2} x^{n-2} - \ldots. + (-1)^n a_0 = 0$
4	Square	$\alpha_1^2, \alpha_2^2, \alpha_3^2 \ldots$	Put $x = \sqrt{x}$	$a_0 x^{n/2} + a_1 x^{(n-1)/2} + a_2 x^{(n-2)/2} + \ldots. + a_n = 0$ & then squaring this equation
5	Cubes	$\alpha_1^3, \alpha_2^3, \alpha_3^3$	Put $x = \sqrt[3]{x}$	$a_0 x^{n/3} + a_1 x^{(n-1)/3} + a_2 x^{(n-2)/3} + \ldots. + a_n = 0$ & then cubing this equation
6	Multiplied by some constant	$k\alpha_1, k\alpha_2, k\alpha_3, \ldots$	Put $x = x/k$	$A_0 (x/k)^n + a_1 (x/k)^{n-1} + a_2 (x/k)^{n-2} + \ldots + a_n = 0$

Therefore if $aX^2 + bX + C = 0$ is the given quadratic equation then the equation

- whose roots are equal in magnitude and **opposite** in sign is $aX^2 - bX + C = 0$
- whose roots are **reciprocals** of roots of the given equation is $cX^2 + bX + a = 0$

Condition for common roots

The equations $a_1X^2 + b_1X + c_1 = 0$ & $a_2X^2 + b_2X + c_2 = 0$ have

- a **common roots** if $(c_1 a_2 - c_2a_1)^2 = (b_1 c_2 - b_2 c_1)(a_1 b_2 - a_2 b_1)$
- have **both common roots** if $a_1/a_2 = b_1/b_2 = c_1/c_2$

Maximum & minimum value of quadratic expressions:

For the quadratic expression $ax^2 + bx + c$

∗ If $a > 0$, the minimum value is $\frac{-D}{4a}$ at $x = \frac{-b}{2a}$

∗ If $a < 0$, the maximum value is $\frac{-D}{4a}$ at $x = \frac{-b}{2a}$

e.g. i. The maximum value of $-X^2 - 5X + 3$ is 37/4.

ii. The minimum value of $X^2 - 6X + 8$ is (-1).

NOTE:

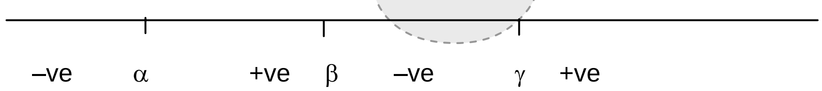
∗ $(X - \alpha)(X - \beta) > 0$ implies X does not lie between α and β.

∗ $(X - \alpha)(X - \beta) < 0$ implies X lie between α and β.

Rule to determine the value of x where f(x) is +ve or –ve:

If $f(x) = (x - \alpha)(x - \beta)(x - \gamma)$

- Put $f(x) = 0$ find value of x i.e. α, β, γ
- Plot the point on integer line in increasing order. Assign +ve & –ve (starting RHS with +ve alternately +ve / –ve) as follows:



so f(x) is +ve when $x > \gamma$ & $\alpha < x < \beta$ and f(x) is –ve when $x < \alpha$ & $\beta < x < \gamma$

NOTE :

∗ *If the sum of two positive quantities is given, their product is greatest when they are equal.*

Ex. Given $X + Y = 30 \Rightarrow$ Possible (X, Y) are (1, 29), (2, 28), (3, 27) …. and so on. Out of all these, the pair that gives the maximum product will be (15, 15)

∗ *If the product of two positive quantities is given, their sum is least when they are equal.*

Ex. Given $X Y = 100 \Rightarrow$ Possible (X, Y) are (1, 100), (2, 50), (4, 25) …… and so on. Out of all these, the pair that gives the minimum sum will be (10, 10).

Complex roots of equations with real coefficients.

If the equation f (x) = 0 with constant coefficient has a complex root $\alpha + i\beta$ ($\alpha, \beta \in R$, $\beta \neq 0$), then the complex conjugate $\alpha - i\beta$ would also be a root of the polynomial equation, f (x) = 0.

Example: Let the equation is $x^2 - 4x + 13 = 0$ Here the coefficient are all real numbers.

$\therefore$ Roots of the equation are $x = \dfrac{4 \pm \sqrt{16 - 52}}{2} = 2 \pm \sqrt{-9}$

$\therefore$ $x = 2 + 3i$ and $x = 2 - 3i$ $\therefore$ The roots are complex conjugate of each other.

Irrational roots of equation with rational coefficients:

If the equation f (x) = 0 with rational coefficients has an irrational root $\alpha + \sqrt{\beta}$ ($\alpha, \beta \in$ Q, $\beta > 0$ and is not a perfect square) then $\alpha - \sqrt{\beta}$ would also be a root of the polynomial equation f (x) = 0

Example: Consider the polynomial equation $x^2 - 4x + 1 = 0$ Here the coefficient are all rational numbers.

$\therefore$ The roots of the equation are $= \dfrac{4 \pm \sqrt{16 - 4}}{2} = \dfrac{4 \pm \sqrt{12}}{2} = \dfrac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$.

$\therefore$ $x = 2 + \sqrt{3}$, $x = 2 - \sqrt{3}$. The irrational roots has occurred in pair

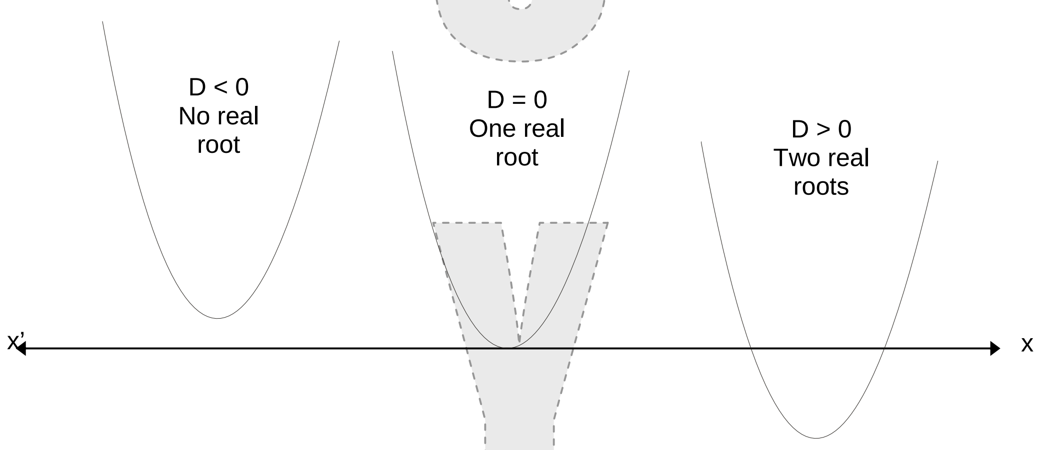
Example: Consider the equation $x^2 - (2 + \sqrt{3})x + 2\sqrt{3}x = 0$ The roots of the equation are 2, $\sqrt{3}$.

Here the roots 2, $\sqrt{3}$ are not in conjugate pair because all the coefficient of the given equation are not integers.

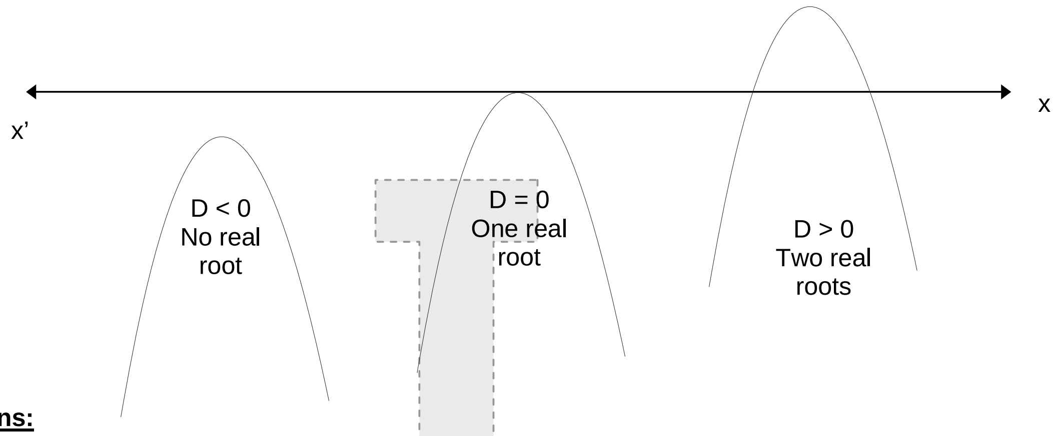
Graph of quadratic equation $y = ax^2 + bx + c$:

Shape of the graph is **parabolic**

Case I: *if a is positive :* Graph is upwards. Different cases arises are



Case II: *if a is negative :* Graph is downwards. Different cases arises are



Deductions:

- If the roots of the equation $ax^2 + bx + c = 0$ are imaginary then the sign of the quadratic expression $ax^2 + bx + c$ is same as that of 'a' for all real values of x.
- The expression $ax^2 + bx + c$ is always positive, if

  D or $b^2 - 4ac < 0$ and $a > 0$.
- The expression $ax^2 + bx + c$ is always negative, if

  $b^2 - 4ac < 0$ and $a < 0$.
- If k_1 & k_2 are the two points such that $f(k_1) \times f(k_2) < 0$, then at least one root lies between k_1 and k_2

Maximum number of positive & negative roots:

- The maximum numbers of positive real roots of polynomial equation f(x) = 0 is the number of changes of signs from positive to negative & negative to positive.
- The maximum numbers of negative real roots of polynomial equation f(x) = 0 is the number of changes of signs from positive to negative & negative to positive in f (– x).

Note: If there is no change in sign in f (x) & f (– x) then there are no real roots i.e. all roots are imaginary.

Eg: Let $f(x) = x^3 + 6x^2 + 11x - 6 = 0$. The sign of various terms are

+ + + –

So, clearly there is only one change of the sign in expression f (x) from +ve to –ve. Therefore f (x) has at the most one +ve real root.

$$f(-x) = -x^3 + 6x^2 - 11x - 6 = 0.$$

– + – –

So, clearly there are two changes of the sign in expression f (x) from +ve to –ve & –ve to +ve. Therefore f (x) have at the most two –ve real roots.

Relation between roots and coefficient of an equation:

Let $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_n$ be the n roots of the equation $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-1} \ldots + a_{n-1} x + a_n = 0$, then we have the following relations.

Sum of the roots taken one at a time = $\sum \alpha_1$ i.e., $\alpha_1 + \alpha_2 + \ldots + \alpha_n = -(a_1/a_0)$.

Sum of the roots taken two at a time = $\sum \alpha_1 \alpha_2 = (a_2/a_0)$

Sum of the roots taken three at a time = $\sum \alpha_1 \alpha_2 \alpha_3 = \alpha_1 \alpha_2 \alpha_3 + \alpha_2 \alpha_3 \alpha_4 + . = -(a_3 / a_0)$

Product of the roots = $\alpha_1 \alpha_2 \alpha_3 \ldots\ldots \alpha_n = \{(-1)^n a_n / a_0\}$.

The expression $\sum \alpha_1$, $\sum \alpha_1 \alpha_2$, ……………………., $\sum \alpha_1 \alpha_2 \alpha_3 \ldots\ldots \alpha_n$ are called the elementary symmetric functions $\alpha_1, \alpha_2, \ldots\ldots \alpha_n$.

Relation for quadratic equations: Let α, β be two roots of the quadratic equation $a_0 x^2 + a_1 x + a_2 = 0$, then $\sum\alpha = \alpha + \beta = -(a_1 / a_0)$, $\sum \alpha\beta = \alpha\beta = (a_2 / a_0)$.

Relation for cubic equations: Let α, β, γ be three roots of the cubic equation $a_0 x^3 + a_1 x^2 + a_2 x + a_3 = 0$, then, $\sum\alpha = \alpha + \beta + \gamma = -(a_1 / a_0)$, $\sum\alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = a_2 / a_0$ $\sum\alpha\beta\gamma = \alpha\beta\gamma = \{-a_3 / a_0\}$

Relation of bi-quadratic equations: Let $\alpha, \beta, \gamma, \delta$ be the four of the bi-quadratic equation.

$$a_0 x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4 = 0, \text{ then}$$

$\sum\alpha = \alpha + \beta + \gamma + \delta = -(a_1 / a_0)$

$\sum\alpha\beta = \alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \beta\delta = a_2 / a_0$

$\sum\alpha\beta\gamma = \alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = -(a_3 / a_0)$

$\sum\alpha\beta\gamma\delta = \alpha\beta\gamma\delta = a_4 / a_0$

Symmetric functions of the roots:

A functions of the roots α, β, γ ……. of an equation is called symmetric when interchanging any two roots in it does not alter it. [A symmetric function is denoted by placing $\sum$ sign before any term of the function.

If α, β, γ be the roots of a cubic equation , then $\sum\alpha\beta = \alpha\beta + \gamma\beta + \alpha\gamma$ $\sum\alpha^2 = \alpha^2 + \beta^2 + \gamma^2$

$\sum\alpha^3 = \alpha^3 + \beta^3 + \gamma^3$ $\sum\alpha\beta^2\gamma^2 = \alpha\beta^2\gamma^2 + \beta\alpha^2\gamma^2 + \gamma\alpha^2\beta^2$ etc. are symmetric functions of α, β, γ

Similarly if $\alpha, \beta, \gamma, \delta$ be the roots of quadratic equation, then

- $\sum\alpha^2\beta = \alpha^2(\beta + \gamma + \delta) + \beta^2(\alpha + \gamma + \delta) + \gamma^2(\alpha + \beta + \delta) + \delta^2(\alpha + \beta + \gamma)$
- $\sum\alpha^2 = (\sum\alpha)^2 - 2\sum\alpha\beta$
- $\sum\alpha^2\beta = \sum\alpha \sum\alpha\beta - 3\alpha\beta\gamma$
- $\sum\alpha^3 = \sum\alpha . \sum\alpha^2 - \sum\alpha^2\beta$.
- $\sum\alpha^2\beta^2 = (\sum\alpha\beta)^2 - 2\alpha\beta\gamma . \sum\alpha$

Ex1. If α, β, γ are the roots of the cubic equation $x^2 + ax^3 + bx + c = 0$, then find the value of the following symmetric functions.

(1) $\sum\alpha^2$ (2) $\sum\alpha^2\beta$ (3) $\sum\alpha^3$

(4) $\sum\alpha^2\beta^2$ (5) $\sum\alpha^2\beta\gamma$ (6) $\sum\alpha^4$

Sol. α, β, γ are the roots of the equation $x^3 + ax^2 + bx + c = 0$

$\therefore \sum\alpha = \alpha + \beta + \gamma = -a$ $\sum\alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = b$ …….(1) $\alpha\beta\gamma = -c$

(1) $(\alpha + \beta + \gamma)^2 = (\alpha^2 + \beta^2 + \gamma) + 2(\alpha\beta + \beta\gamma + \gamma\alpha)$ or $(\sum\alpha)^2 = \sum\alpha^2 + 2\sum\alpha\beta$

$\therefore \sum\alpha^2 = (\sum\alpha)^2 - 2\sum\alpha\beta = (-a)^2 - 2b = a^2 - 2b$

(2) $\sum\alpha \sum\alpha B = \sum\alpha^2\beta + 3\alpha\beta\gamma$

$\therefore \sum\alpha^2\beta = \sum\alpha \sum\alpha\beta - 3\alpha\beta\gamma = (-a)(b) - 3(-c) = 3c - ab$

(3) $(\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2) = (\alpha^3 + \beta^3 + \gamma^3) + (\alpha\beta^2 + \alpha\gamma^2 + \beta\alpha^2 + \beta\gamma^2 + \gamma\alpha^2 + \gamma\beta^2)$

or, $\sum\alpha \sum\alpha^2 = \sum\alpha^3 + \sum\alpha^2\beta \;\therefore \sum\alpha^3 = \sum\alpha.\sum\alpha^2 - \sum\alpha^2\beta = (-a)(a^2 - 2b) - (3c - ab)$

$= 3ab - a^3 - 3c$

(4) $(\alpha\beta + \beta\gamma + \gamma\alpha)^2 = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2) + 2\alpha\beta\gamma(\alpha + \beta + \gamma)$ or, $(\sum\alpha\beta)^2 = \sum\alpha^2\beta^2 + 2\alpha\beta\gamma.\sum\alpha$

$\therefore \sum\alpha^2\beta^2 = (\sum\alpha\beta)^2 - 2\alpha\beta\gamma.\sum\alpha = b^2 - 2(-c)(-a) = b^2 - 2ac.$

(5) $\sum\alpha^2\beta\gamma = \alpha^2\beta\gamma + \gamma\alpha\beta^2 + \gamma^2\alpha\beta = \alpha\beta\gamma(\alpha + \beta + \gamma) = (-c)(-a) = ac.$

(6) $(\alpha^2 + \beta^2 + \gamma^2)^2 = (\alpha^4 + \beta^4 + \gamma^4) + 2(a^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2)$ or, $(\sum\alpha^2)^2 = \sum\alpha^4 + 2\alpha^2\beta^2$

$= (a^2 - 2b)^2 - 2(b^2 - 2ac)$ [from (1) and (4)] $= (a^4 + 4b^2 - 4a^2b) = (2b^2 - 4ac)$

$= a^4 + 2b^2 + 4ac - 4a^2b.$

Synthetic Division:

This method is used to find the remainder & quotient when a polynomial is divided by $(x - \alpha)$

For example: Let $f(x) = 5x^4 + 3x^3 - 2x^2 + 4x + 7$ we want to divide it by $(x - 2)$, write down of coefficients of powers of x as

Put x – 2= 0	x^4	x^3	x^2	x	7
➢ x = 2	5	3	– 2	4	7
2		10	26	48	104
2 ×	5	13	24	52	111
	X^3	x^2	x	Constant	Reminder

So $\frac{5x^4 + 13x^3 - 2x^2 + 4x + 7}{(x - 2)} = 5x^3 + 13x^2 + 24x + 52 + \frac{111}{x - 2}$

⇒ Put $(x - \alpha)$ i.e. $x - 2 = 0$, $x = 2$

⇒ Write down the coefficient of x in order of their descending power as shown above

⇒ Note down the coefficient of highest power as it and then multiply it by α i.e. 2., and add to the next degree coefficient and so on.